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SOME REMARKS ON THE SIMULTANEOUS CHROMATIC NUMBER

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We present several partial results, variants, and consistency results concerning the following (as yet unsolved) conjecture. If X is a graph on the ground set V with $\mathrm{Chr}(X) = \aleph_1$ then X has an edge coloring F with $\aleph_1$ colors such that if V is decomposed into $\aleph_0$ parts then there is one in which F assumes all values.

In this paper we consider a somewhat technical statement on uncountably chromatic graphs. In order to formulate our results we let $P(X,\lambda,\kappa)$ denote that X is a graph on some vertex set V and there is a function $F : X \to \lambda$ with the property that whenever $V = \bigcup\{V_i : i < \mu\}$ is a decomposition of the vertex set into $\mu < \kappa$ parts then there exist some $i < \mu$ that F assumes every value on $X \cap [V_i]^2$. $Q(\kappa)$ is the abbreviation of the statement that $P(X,\kappa,\kappa)$ holds for every graph X with chromatic number κ. This concept, which is interesting on its own right, arises naturally when one wants to construct large chromatic triple systems avoiding certain predetermined finite configurations. This is the reason why it was introduced and investigated in [1] where considerable efforts were made to describe those triple systems appearing in every uncountably chromatic triple system. The authors of [1] observed with surprise that they could not prove $Q(\aleph_1)$ even for the complete graph on $\aleph_1$ vertices (in ZFC). This was proved years later by Todorčević and his idea eventually led him to prove his famous result;

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Due to some unfortunate misunderstandings, this paper appeared much later than we expected.

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$\omega_1 \not\to [\omega_1]^2_{\omega_1}$ (see [8]). Although it seems very hard to establish $Q(\aleph_1)$ even for very simple graphs all evidence suggests that it is just true in ZFC. Unable to prove this conjecture in general, in this paper we prove some weaker results concerning it.

First we show that if $\kappa^{<\kappa}=\kappa$ and a Cohen subset is added to κ then, in the resulting model, $Q(\kappa^+)$ holds (Theorem 1). If, for some κ, more than κ Cohen subsets are added to μ, then, in the resulting model for every graph X of cardinal κ with chromatic number greater than μ $P(X,\mu,\mathrm{Chr}(X))$ holds (Theorem 2). These results do not yet give the consistency of $P(X,\kappa,\kappa)$ for strongly inaccessible κ. In Theorem 3 we prove that this is consistent, at least, for a wide class of graphs.

Next we prove $Q(\aleph_1)$ for graphs of size $\aleph_1$ if the principle $\diamondsuit^+$ holds. In fact we prove the stronger statement that if X is an uncountably chromatic graph on ω_1 then there is a coloring $F:X\to\omega_1$ of the edge set that if $A\subseteq\omega_1$ induces an uncountably chromatic subgraph on X then F assumes every value on A. We also give the consistency of the negation of this latter statement. Unfortunately, our method does not give a proof of the independence of $Q(\aleph_1)$. It is however, consistent (from a measurable cardinal) that for some weakly inaccessible κ there is X, a graph on κ with chromatic number κ such that whenever $F:X\to\omega_1$ is an edge-coloring, then there is a countable decomposition $\kappa = V_0\cup V_1\cup\cdots$ of the vertices with all V_i inducing just countably many colors (Theorem 6).

Although we have not been able to show that $Q(\aleph_1)$ holds we prove a result which is equally useful for the construction of large chromatic triple systems avoiding some finite substructures. Namely, we prove (Theorem 7) that for every uncountably chromatic graph X and cardinal τ there is a graph Y with an edge-coloring of τ colors, such that when the vertices of Y are colored with less than $\mathrm{Chr}(X)$ colors, then, in some color class, all edge-colors occur, and Y has the same collection of finite induced subgraphs as X, in short, $\mathrm{FS}(Y)=\mathrm{FS}(X)$. We notice that not much is known about the collections of finite subgraphs which occur as $\mathrm{FS}(X)$ for some κ-chromatic graph X (where $\kappa>\omega$). It is not known, e.g, that the answer is independent of κ (Taylor's conjecture). (We notice that since the submission of the manuscript Komjáth, using a model of Shelah proved the independence of Taylor's conjecture).

Further, we show that if λ is a singular cardinal, $\mu<\lambda$ then there is a coloring $F:[\lambda^+]^2\to\mu$ with the property that if λ^+ is decomposed into λ parts then in one of them F assumes all values (that is, $P([\lambda^+]^2,\mu,\lambda^+)$ holds for every $\mu<\lambda$).

Although—as we mentioned—it is not too easy to prove $Q(\aleph_1)$ for the complete graph on $\aleph_1$ vertices, assuming the Continuum Hypothesis one can show this via the usual diagonal argument. This can easily been transformed into a proof that there is a coloring $[2^\omega]^2$ with countably many colors such that if 2^ω is decomposed into countably many parts, one of them gets all the colors. In our last Theorem we generalize this for the shift-graphs, we prove that $P(\mathrm{Sh}_n(\exp_n(\kappa)), \kappa, \kappa^+)$ holds if $2 \le n < \omega$ and κ is an infinite cardinal.

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Definitions. Notation. We follow the established axiomatic set theory convention on notation and notions. If κ is an infinite cardinal then $\exp_0(\kappa) = \kappa$, and $\exp_{n+1}(\kappa) = 2^{\exp_n(\kappa)}$ for $n = 0, \ldots$.

If S is a set, κ a cardinal then we let

$$[S]^\kappa = \{x \subseteq S : |x| = \kappa\}, [S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}.$$

A *graph* is a pair (V, X) where V is some set (the set of vertices) and $X \subseteq [V]^2$ (the set of edges). We sometimes just write X rather than (V, X). If (V, X) is a graph then $A \subseteq V$ is *independent* iff $X \cap [A]^2 = \emptyset$. If (V, X) is a graph $A \subseteq V$ then $(A, X \cap [A]^2)$ is the restriction of X to A, denoted by $X|A$. Graphs obtained by restriction are the *induced subgraphs of* (V, X). For a graph X let $\mathrm{FS}(X)$ denote the set of all finite induced subgraphs of X.

The *chromatic number* of a graph $= (V, X)$, $\mathrm{Chr}(V, X)$ is the minimal cardinal μ for which there is a function $f : V \to \mu$ with the property that $f(x) \neq f(y)$ holds whenever x and y are joined in (V, X).

For $2 \le n < \omega$, $\lambda \ge \omega$ *the n-shift graph on* λ, $\mathrm{Sh}_n(\lambda)$ is the following. Its vertex set is $[\lambda]^n$, and the edges are of the pairs of the form

$$\{\{x_0, \ldots, x_{n-1}\}, \{x_1, \ldots, x_n\}\}$$

where $x_0 < x_1 < \cdots < x_n < \lambda$. In [2] it is shown that $\mathrm{Chr}(\mathrm{Sh}_{n+1}(\lambda)) \le \kappa$ iff $\lambda \le \exp_n(\kappa)$.

$P(X, \lambda, \kappa)$ denotes the following statement. X is a graph on some vertex set V and there is a function $F : X \to \lambda$ with the following property. Whenever $V = \bigcup\{V_i : i < \mu\}$ is a decomposition of the vertex set into $\mu < \kappa$ parts then there exist some $i < \mu$ that F assumes every value on $X \cap [V_i]^2$.

$Q(\kappa)$ is the abbreviation of the statement that $P(X, \kappa, \kappa)$ holds for every graph (V, X) with $\mathrm{Chr}(V, X) = \kappa$.

Theorem 1. *Assume that $\kappa^{<\kappa} = \kappa$. If we add a Cohen-generic subset to κ then, in the resulting model, $Q(\kappa^+)$ holds.*

Proof. We force with the following partial ordering. $p \in P$ iff p is a function, $\mathrm{Dom}(p) \in \kappa$ (so $\mathrm{Dom}(p)$ is an initial segment of the ordered set κ), $\mathrm{Ran}(p) \subseteq \kappa$. $p \leq q$ iff p extends q.

Clearly, forcing with $(P,<)$ adds a function $G:\kappa \to \kappa$. We notice that for every formula φ of the forcing language which is true in the generic model $V[G]$ there is a unique shortest condition $p \in G$ which forces φ.

We fix the functions $\{f_\gamma : \kappa^+ \to \kappa^+, \gamma < \kappa\}$ with the property that for $\alpha < \kappa^+$ we have $\alpha \subseteq \{f_\gamma(\alpha) : \gamma < \kappa\}$.

Assume that X is a graph in $V[G]$ with chromatic number κ^+. We can assume that there is a cardinal λ such that 1 forces that the vertex set of X is λ and also that $\lambda = \bigcup\{A_\alpha : \alpha < \kappa^+\}$ where every set A_α is independent.

We define $F: X \to \kappa^+$ as follows. Assume that $\{x,y\} \in X$, $x \in A_\beta$, $y \in A_\alpha$ for some $\beta < \alpha$. There is a $\gamma < \kappa$ such that $\beta = f_\gamma(\alpha)$.

Let $\delta \geq \gamma$ be minimal that $G|\delta$ forces $x \in A_\beta$, $y \in A_\alpha$, and $\{x,y\} \in X$. Assume that $\tau = G(\delta)$. Set $F(x,y) = f_\tau(\alpha)$.

In order to show $Q(\kappa^+)$ assume that 1 forces that λ is decomposed as

$$\lambda = \bigcup\{B_i : i < \kappa\}.$$

For every $x < \lambda$ there are $p(x) \in G$ and $i(x) < \kappa$ such that $p(x) \Vdash x \in B_{i(x)}$. The mapping $x \mapsto (p(x), i(x))$ is a κ-coloration of λ, in $V[G]$. As $\mathrm{Chr}(X) = \kappa^+$ there is some pair (p,i) such that the set

$$L = \{x < \lambda : p(x) = p, i(x) = i\}$$

is κ^+-chromatic. We claim that F assumes every value on B_i. Let $\xi < \kappa^+$ be arbitrary. As L is κ^+-chromatic there is some $\alpha > \xi$ such that the set

$$K = \{\beta < \alpha : \exists x \in A_\beta \cap L, \exists y \in A_\alpha \cap L, \{x,y\} \in X\}$$

has cardinal κ. Assume that $\gamma = \mathrm{Dom}(p)$ and pick an element $\beta \in K$ with $\beta = f_\delta(\alpha)$ for some $\delta \geq \gamma$. Select also $x \in A_\beta \cap L$, $y \in A_\alpha \cap L$ with $\{x,y\} \in X$. Let $p' \in G$ be the shortest condition with $\mathrm{Dom}(p') \geq \delta$, forcing $x \in A_\beta$, $y \in A_\alpha$, and $\{x,y\} \in X$. If $\mathrm{Dom}(p') = \delta'$, then we can set $p'' \leq p'$ where $\mathrm{Dom}(p'') = \delta' + 1$ and $p''(\delta') = \tau$, here $\tau < \kappa$ is an ordinal for which $f_\tau(\alpha) = \xi$ holds. Now p'' forces that $F(x,y) = \xi$. ∎

Theorem 2. *($\mu = \mu^{<\mu}$) If more than κ μ-Cohen sets are added, then in the resulting model the following is true. If X is a τ-chromatic graph on κ, $\tau > \mu$, then $P(X,\mu,\tau)$ holds.*

Proof. Let P be the forcing notion that adds $\lambda > \kappa$ μ-Cohen sets. That is, $p \in P$ iff p is a function, $\mathrm{Dom}(p) \in [\lambda]^{<\mu}$, $\mathrm{Ran}(p) \subseteq \mu$. $p \leq q$ holds if $p \supseteq q$. Let X be a graph as described in the statement of the Theorem. We can assume, with no harm, that $X \in V$ (the ground model), and that $2^\mu \geq \kappa$ holds in V. Let $\pi : X \to \kappa$ be a bijection between the edges of X and κ. The forcing P adds a generic function $G : \kappa \to \mu$. We set $F(e) = G(\pi(e))$ for $e \in X$. We show that this F establishes $P(X, \mu, \tau)$ in V^P.

Assume the contrary, that is that in V^P the following holds. There is a decomposition $\kappa = \bigcup\{V_i : i < \varphi\}$ with $\varphi < \mu$ and some values $j(i) < \mu$ (for $i < \varphi$) such that F misses $j(i)$ on $X \cap [V_i]^2$.

Select, for each $\alpha < \kappa$ a condition $p_\alpha \in P$ determining an i with $\alpha \in V_i$ and the value of $j(i)$,

$$p_\alpha \Vdash \alpha \in V_{i(\alpha)}, j(i) = k(\alpha)$$

for some $i(\alpha) < \varphi$ and $k(\alpha) < \mu$.

We now define a set mapping on κ. For $\alpha < \kappa$ let $\beta \in g(\alpha)$ whenever $\beta \neq \alpha$ and there is some $e \in X$ with $\beta \in e$, $\pi(e) \in \mathrm{Dom}(p_\alpha)$. Clearly, $|g(\alpha)| < \mu$. By Fodor's set mapping theorem (see e.g., in [4]) κ can be decomposed into the union of μ free sets, that is, with $\alpha \notin g(\beta)$ if α, β are in the same set. We also notice that as $\kappa \leq 2^\mu$ the system $\{p_\alpha : \alpha < \kappa\}$ is μ-centered. With these in mind, we can define a decomposition $\kappa = \bigcup\{W_\xi : \xi < \varphi\}$ such that for $\alpha, \beta \in W_\xi$ we have $i(\alpha) = i(\beta)$, $k(\alpha) = k(\beta)$, $\beta \notin g(\alpha)$, and p_α, p_β are compatible.

As $\mathrm{Chr}(X) > \varphi$ holds, there are some $\alpha, \beta \in W_\xi$ which are joined. Now $p_\alpha \Vdash \alpha \in V_i, j(i) = k$ and $p_\beta \Vdash \beta \in V_i, j(i) = k$ for some i, k. Let $e = \{\alpha, \beta\} \in X$. As $\pi(e) \notin \mathrm{Dom}(p_\alpha) \cup \mathrm{Dom}(p_\beta)$ and p_α, p_β are compatible, there is some $q \leq p_\alpha, p_\beta$ with $q(\pi(e)) = k$, that is, forcing $F(e) = k$, a contradiction. ∎

Theorem 3. *(GCH) If κ is strongly inaccessible and more than κ Cohen-subsets are added to κ then the following will be true in the resulting model. Let X be a κ-chromatic graph on κ for which $\sup\{\mathrm{Chr}(X|\alpha) : \alpha < \kappa\} < \kappa$. Then $P(X, \kappa, \kappa)$ holds.*

Proof. We can assume, with no loss of generality, that X is in V, the ground model, and we add just one Cohen-generic function, $G : \kappa \to \kappa$. Fix , in V, a bijection $\pi : X \to \kappa$ and define the edge coloring of X, in $V[G]$ as $F(e) = G(\pi(e))$. We show that this F witnesses that $P(X, \kappa, \kappa)$ holds. By a theorem of Shelah's (see [6]) there is a function $f : \kappa \to \kappa$ with $\alpha < f(\alpha)$ for $\alpha < \kappa$ such that whenever $C \subseteq \kappa$ is a closed, unbounded set then X restricted to $\bigcup\{[\alpha, f(\alpha)) : \alpha \in C\}$ is κ-chromatic. Fix such an f. Let D be a closed, unbounded set such that $\alpha < \gamma \in D$ implies that $f(\alpha) < \gamma$.

Let Y be the following subgraph of X: for $\{x,y\} \in X$, $x < y$, the edge $\{x,y\}$ is in Y if and only if it is separated by some element of D. By our assumption, the graph $X - Y$ is the vertex disjoint union of graphs with chromatic numbers bounded below κ, therefore $\mathrm{Chr}(Y) = \kappa$.

Assume that some condition p forces that a function $h : \kappa \to \mu$ gives a decomposition witnessing the failure of F for $P(X,\kappa,\kappa)$, that is, if $i < \mu$, then some $j(i) < \kappa$ is not in the range of F restricted to $[h^{-1}(i)]^2$. We can as well assume that p determines μ and the values $j(i)$.

By transfinite recursion on $\alpha < \kappa$ define the decreasing, continuous sequence of conditions $\{p_\alpha : \alpha < \kappa\}$ with the following properties. Set $\gamma_\alpha = \mathrm{Dom}(p_\alpha)$. Set $p_0 = p$. Given p_α choose $p_{\alpha+1} \le p_\alpha$ in such a way that if $y < \gamma_\alpha \le x < f(\gamma_\alpha)$, $p_\alpha \Vdash h(y) = i$, and $\pi(\{y,x\}) \ge \gamma_\alpha$ then

$$p_{\alpha+1} \Vdash F(y,x) = j(i)$$

(clearly we can achieve this) and $p_{\alpha+1}$ determines h in the interval $[\gamma_\alpha, f(\gamma_\alpha))$.

Having finished the construction of the sequence $\{p_\alpha : \alpha < \kappa\}$ let $C \subseteq D$ be a closed, unbounded sequence that $\alpha = \gamma_\alpha$ holds for $\alpha \in C$.

By the property of function f there is an $\alpha \in C$ such that $h(y) = h(x) = i$ holds for some $y < \alpha \le x < f(\alpha)$ and $p_{\alpha+1} \Vdash F(y,x) = j(i)$ which is a contradiction. ∎

Theorem 4. *($\diamondsuit^+$) If X is a graph on ω_1 with $\mathrm{Chr}(X) = \aleph_1$ then there is a function $F : X \to \omega_1$ with the property that if $A \subseteq \omega_1$ induces a subgraph with uncountable chromatic number then F assumes every value on $X \cap [A]^2$. (So $P(X,\omega_1,\omega_1)$ holds.)*

Proof. By $\diamondsuit^+$ there are sets $\{S_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ such that if $A \subseteq \omega_1$ then there is a closed unbounded set $C \subseteq \omega_1$ with the property that for every $\gamma \in C$ there is an $n < \omega$ with $A \cap \gamma = S_{\gamma,n}$.

Using this system we are going to define the required function F. At step α we color the edges of X going down from α. Consider for $\beta \le \alpha$, $n < \omega$ those sets

$$T_{\beta,n} = \{x \in S_{\beta,n} : \{x,\alpha\} \in X\}$$

which are infinite. By a diagonal argument we can define the values

$$\{F(x,\alpha) : x < \alpha, \{x,\alpha\} \in X\}$$

in such a way that

$$\{F(x,\alpha) : x \in T_{\beta,n}\} = \alpha$$

always holds whenever $T_{\beta,n}$ is infinite.

In order to show that this is a good coloring assume that $A\subseteq\omega_1$ is a subset that misses a certain color ξ. Let $C\subseteq\omega_1$ be a closed, unbounded set as in the statement of $\diamondsuit^+$, for the set A. Assume that $\alpha\in A$, $\alpha>\xi$, and $\gamma\leq\alpha<\gamma'$ where γ, γ' are two successive elements of C. For some $n<\omega$, $A\cap\gamma=S_{\gamma,n}$. The fact that no edge of $X\cap[A]^2$ gets color ξ implies (by the construction) that only finitely many edges go from α to $S_{\gamma,n}=A\cap\gamma$. In other words, C splits ω_1 into countable sets such that from each point in A only finitely many edges go into the union of the previous sets. But this implies that the chromatic number of $X\cap[A]^2$ is countable. ∎

We prove the independence of this latter statement.

Theorem 5. *It is consistent that $2^{\aleph_0}=\aleph_2$ and there is a graph X on ω_1 with $\mathrm{Chr}(X)=\aleph_1$ and for every function $F:X\to 2$ there is some $\aleph_1$-chromatic $A\subseteq\omega_1$ where F assumes only one value.*

Proof. We show that a graph X on ω_1 can exist which is uncountably chromatic on every stationary set and if $F:X\to 2$ then there is some stationary $A\subseteq\omega_1$ on which F is constant. This clearly suffices.

Let V be a model of the continuum hypothesis in which $\{T_\alpha:\alpha<\omega_2\}$ are stationary sets in ω_1 with pairwise countable intersection. (For example, $\diamondsuit$ implies the existence of sets like these.)

We define a ccc iterated notion of forcing of length ω_2. In step 0 we add a graph on ω_1 with finite conditions, this will be X. In more details, $q\in Q_0$ iff q is of the form $q=(s,g)$ where $s\in[\omega_1]^{<\omega}$ and $g\subseteq[s]^2$, that is, g is a graph on the finite set s. $q'=(s',g')\leq q=(s,g)$ iff $s'\supseteq s$ and $g=g'\cap[s]^2$.

If $G_0\subseteq Q_0$ is a generic subset then we let $X=\bigcup\{g:(s,g)\in G_0\}$ be the generic graph. We are going to construct a finite support iteration of length ω_2. Assume that we arrived at stage $\alpha<\omega_2$. Consider a function $F_\alpha:X\to\{0,1\}$. We set $i_\alpha=0$ or 1 according to if there is a stationary subset T' of T_α on which F_α only assumes the value 0. If $i_\alpha=0$ we let T'_α be such a set. If $i_\alpha=1$ we set $T'_\alpha=\bigcup\{p(\{\alpha\}):p\in G\}$. If $i_\alpha=0$ we do nothing, i.e., Q_α is the trivial forcing. If $i_\alpha=1$ we force with the finite, homogeneous, 1-colored sets, i.e., $q\in Q_\alpha$ iff $q\in[T_\alpha]^{<\omega}$, and $F_\alpha(x,y)=1$ holds for $\{x,y\}\in X\cap[g]^2$.

As we indicated above $P_{\alpha+1}=P_\alpha\star Q_\alpha$ and we use direct limits at limit steps. For $p\in P_\alpha$, let $\mathrm{supp}(p)$ be the support of p, that is, $\beta<\alpha$ is in it unless $p|\beta\Vdash p(\beta)=1$. $\mathrm{supp}(p)$ is always a finite subset of α.

We call a condition $p\in P_\alpha$ *determined* iff for every $\beta\in\mathrm{supp}(p)$, $\beta>0$, $p|\beta$ determines the values of i_β, $p(\beta)$, and if $i_\beta=1$, $p(0)=(s,g)$ then $p(\beta)\subseteq s$ holds. Let D_α denote the set of determined conditions.

Lemma 1. *For every $\alpha\leq\omega_2$ D_α is dense in P_α.*

Proof. By induction on α. The cases when $\alpha = 0$ or limit are obvious. Assume that $(p,q) \in P_{\alpha+1} = P_\alpha \star Q_\alpha$. Choose $p' \leq p$ that determines i_α and the finite set q. Extend p' to $p'' \in D_\alpha$ (possible by the inductive hypothesis). Finally, extend p'' further to r such that the vertex set of the graph $r(0)$ should include q. Then $(r,q) \leq (p,q)$ is in $D_{\alpha+1}$. ∎

Lemma 2. *For every $\alpha \leq \omega_2$ P_α is ccc.*

Proof. Assume that $p_\xi \in P_\alpha$ for $\xi < \omega_1$. By the previous Lemma $p_\xi \in D_\alpha$ can be assumed. By the Δ-system lemma and the pigeon-hole principle we can assume that the supports form a Δ-system; $\mathrm{supp}(p_\xi) = B \cup A_\xi$, $B = \{\beta_1, \dots, \beta_n\}$, $p_\xi(0) = (s \cup s_\xi, g_\xi)$ with $g_\xi \cap [s]^2$ independent of ξ, and we can even assume that for any $1 \leq j \leq n$ the sets $p_\xi(\beta_j) \cap s$ are the same. Now select $\xi \neq \xi'$. Define q as follows. $\mathrm{supp}(q) = B \cup A_\xi \cup A_{\xi'}$. $q(0) = (s \cup s_\xi \cup s_{\xi'}, g_\xi \cup g_{\xi'})$ and for $0 < \gamma < \alpha$

$$q(\gamma) = \begin{cases} p_\xi(\gamma) & (\gamma \in A_\xi) \\ p_{\xi'}(\gamma) & (\gamma \in A_{\xi'}) \\ p_\xi(\gamma) \cup p_{\xi'}(\gamma) & (\gamma \in B - \{0\}). \end{cases}$$

It is easy to see that q is a condition and it is a common extension of p_ξ and $p_{\xi'}$. ∎

Standard arguments with CH and the above Lemma give that with an appropriate bookkeeping we can make sure that in $V^{P_{\omega_2}}$ the sequence $\{F_\alpha : \alpha < \omega_2\}$ enumerates all functions from X to $\{0,1\}$. Also, if $i_\alpha = 0$ for some $\alpha < \omega_2$ then T'_α will stay stationary in the final model.

Lemma 3. *For $\alpha < \omega_2$, T'_α is stationary.*

Proof. By ccc, it suffices to show that $T'_\alpha \cap C \neq \emptyset$ for every closed, unbounded subset $C \subseteq \omega_1$ which is an element of V. Assume that $(p,q) \in D_{\alpha+1}$ and p forces $i_\alpha = 1$. If $p(0) = (s,g)$ select $\xi \in (C \cap T_\alpha) - s$, and extend p to p' by setting $p'(0) = (s \cup \{\xi\}, g)$, $p'(\beta) = p(\beta)$ for $0 < \beta < \alpha$, and $q' = q \cup \{\xi\}$. Then clearly $(p',q') \leq (p,q)$ forces $\xi \in T'_\alpha \cap C$. ∎

Lemma 4. *In $V^{P_{\omega_2}}$, X has an edge in every stationary set.*

Proof. Assume that $1 \Vdash \underset{\sim}{S}$ is stationary. Then there is a stationary set S and there are conditions $\{p_\xi : \xi \in S\}$ such that $p_\xi \Vdash \xi \in S$.

By repeated applications of the pressing down lemma we can assume that the following hold. For $\xi \in S$ $\mathrm{supp}(p_\xi) = B \cup A_\xi$, a Δ-system. $p_\xi(0) = (s \cup s_\xi, g_\xi)$ with $\xi \in s_\xi$ and $g_\xi \cap [s]^2 = g$ for some $g \subseteq [s]^2$. For $\beta \in B - \{0\}$ the coordinates

$p_\xi(\beta)$ again form a Δ-system; $p_\xi(\beta)=q_\beta\cup q_\xi(\beta)$. As the sets $\{T_\alpha:\alpha<\omega_2\}$ are almost disjoint we can further assume that $S\cap T_\beta=\emptyset$ holds for all but at most one $\beta\in B-\{0\}$.

Assume that there is such an exceptional β and indeed $S\subseteq T_\beta$ (the other case is simpler).

Assume first that there are $\xi\neq\eta$ in S such that $\overline{p}(\xi,\eta)$ does not force $F_\beta(\xi,\eta)=0$ where $\overline{p}(\xi,\eta)\in P_\beta$ is the following condition.

$$\overline{p}(\xi,\eta)(0)=(s\cup s_\xi\cup s_\eta, g_\xi\cup g_\eta\cup\{\{\xi,\eta\}\})$$

and for $0<\gamma<\beta$

$$\overline{p}(\xi,\eta)(\gamma)=\begin{cases} p_\xi(\gamma) & (\gamma\in A_\xi\cap\beta)\\ p_\eta(\gamma) & (\gamma\in A_\eta\cap\beta)\\ p_\xi(\gamma)\cup p_\eta(\gamma) & (\gamma\in(B\cap\beta)-\{0\}).\end{cases}$$

Notice that $\overline{p}(\xi,\eta)$ is indeed a condition as no coordinate $0<\gamma<\beta$ of it contains *both* ξ and η (if $\gamma\in A_\xi$ then $\overline{p}(\xi,\eta)(\gamma)$ may only contain ξ,if $\gamma\in A_\eta$ then $\overline{p}(\xi,\eta)(\gamma)$ may only contain η, and if $\gamma\in B$ then $\overline{p}(\xi,\eta)(\gamma)$ contains neither).

Now extend $\overline{p}$ to $\overline{\overline{p}}\in P_\beta$ such that $\overline{\overline{p}}\Vdash F_\beta(\xi,\eta)=1$. And finally consider p^* where $p^*|\beta=\overline{\overline{p}}$, and otherwise

$$p^*(\gamma)=\begin{cases} p_\xi(\gamma) & (\gamma\in A_\xi-\beta)\\ p_\eta(\gamma) & (\gamma\in A_\eta-\beta)\\ p_\xi(\gamma)\cup p_\eta(\gamma) & (\gamma\in B-\beta).\end{cases}$$

Once again, p^* is a condition (as we arranged things just right at the critical coordinate β) and forces an edge of X (namely $\{\xi,\eta\}$) in $\underline{S}$.

Assume finally that whenever $\xi\neq\eta$ are in S then $\overline{p}(\xi,\eta)\Vdash F_\beta(\xi,\eta)=0$. Set $p'_\xi=p_\xi|\beta$. Define $p\in P_\beta$ as follows. $\mathrm{supp}(p)=B\cap\beta$, $p(0)=(s,g)$ and $p(\gamma)=q_\gamma$ for $\gamma\in(B\cap\beta)-\{0\}$. Notice that $p'_\xi\le p$ for every $\xi\in S$ and if $p'\le p$ then p' is compatible with all but finitely many p'_ξ.

We claim that $p\Vdash i_\beta=0$. Let G_β be a generic set with $p\in G_\beta$ and define $U=\{\xi:p'_\xi\in G_\beta\}$. We show that U witnesses that $i_\beta=0$. Indeed, if p'_ξ, $p'_\eta\in G_\beta$ and $\{\xi,\eta\}\in X$ then $\overline{p}(\xi,\eta)\in G_\beta$ and so we have that $F(\xi,\eta)=0$ as claimed. To show that U is stationary assume that $p'\le p$ and C is closed, unbounded. By our above remarks there is some $\xi\in C\cap S$ such that p' and p'_ξ are compatible. We therefore proved that p forces that Q_β is the trivial forcing. In this case for any $\xi\neq\eta$ in S we can take $\overline{p}(\xi,\eta)$, it forces an edge of X in $\underline{S}$ and we are done. ∎

As Lemma 4 implies that X is uncountably chromatic on every stationary set we are finished. ∎

Theorem 6. *If the existence of a measurable cardinal is consistent then it is consistent that for some inaccessible κ there is a graph X on κ with* $\mathrm{Chr}(X)=\kappa$ *and if $F:X\to\omega_1$ then there is some countably chromatic $A\subseteq\kappa$ such that*

$$\sup\{F(x,y) : x,y\notin A\}<\omega_1.$$

Proof. We borrow the model of [5]. Let V be a model in which κ is a measurable cardinal, I a κ-complete, normal ideal on κ.

First we add a graph on κ with finite conditions. That is, $p\in P$ iff p is of the form $p=(s,g)$ where $s\in[\kappa]^{<\omega}$, $g\subseteq[s]^2$. $p'=(s',g')\leq p=(s,g)$ iff $s'\supseteq s$ and $g=g'\cap[s]^2$. If $G\subseteq P$ is a generic filter, set $X=\bigcup\{g:(s,g)\in G\}$.

In the next step, for every $A\in I$ (if A is infinite) we force with the forcing notion Q_A which makes X countably chromatic on A. That is, Q_A is defined in V^P as follows. $f\in Q_A$ iff f is a function, $\mathrm{Dom}(f)\in[A]^{<\omega}$, and $\{x,y\}\in X$ implies $f(x)\neq f(y)$. $f'\leq f$ iff f' extends f.

Let Q be the finite support product of these notions of forcing. Our final model is $V^{P\star Q}$. In [5] it is shown that $P\star Q$ is ccc, in $V^{P\star Q}$ $\mathrm{Chr}(X)=\kappa$ yet X is countably chromatic on every $A\in I$.

It is now easy to deduce the property described in the Theorem. Assume that $1\Vdash F:X\to\omega_1$. By ccc, there is in V a function $H:[\kappa]^2\to\omega_1$ that $1\Vdash F(\xi,\eta)<H(\xi,\eta)$ for $\xi<\eta<\kappa$. By the Rowbottom partition theorem there is some $A\in I$ such that H is bounded off A, and we are done as X is countably chromatic on A. ∎

We recall that $\mathrm{FS}(X)$ denotes the set of all finite induced subgraphs of X.

Theorem 7. *If* $\mathrm{Chr}(X)=\kappa>\omega$, *$\tau$ is a cardinal then there is a graph Y such that* $\mathrm{FS}(Y)=\mathrm{FS}(X)$ *and $P(Y,\tau,\kappa)$ hold.*

Proof. We first consider the case when κ is successor, $\kappa=\mu^+$. First we are going to create a Y as required with the weaker property $\mathrm{FS}(Y)\subseteq\mathrm{FS}(X)$. We can assume that $\lambda=|V(X)|$ is minimal, i.e., if $X'\subseteq X$ has $|V(X')|<\lambda$ then $\mathrm{Chr}(X')<\kappa$.

Decompose $V(X)$ as the increasing, continuous union of elementary submodels of smaller cardinality, $V(X)=\bigcup\{V_\alpha:\alpha<\lambda\}$, $|V_\alpha|<\lambda$. Assume, for simplicity's sake, that $V_0=\emptyset$, $V(X)=\lambda$, and each V_α is an ordinal, $V_\alpha=\delta_\alpha$ (so $\delta_0=0$).

Assume that s is a finite subset of δ_ω, more exactly, $s=s^0\cup\cdots\cup s^k$ with $s^i\subseteq[\delta_i,\delta_{i+1})$.

Set $Z=X|s$. Let $n>1$ be a natural number. We are going to define $\mathcal{Z}^n$, the class of some duplications of Z. Let the finite sets

$$\{\overline{s}^0,\overline{s}^1_{j_1},\ldots,\overline{s}^k_{j_1\cdots j_k} : 0\leq j_1,\ldots,j_k\leq n\}$$

be pairwise disjoint. Assume that they possess the bijections $\pi_{j_1\cdots j_r}: s^r \to \overline{s}^r_{j_1\cdots j_r}$. A graph Z' is in $\mathcal{Z}^n$ iff $\pi_{j_1\cdots j_r}$ is an isomorphism between $Z|s^r$ and $Z'|\overline{s}^r_{j_1\cdots j_r}$, moreover, if $r' < r$, $x' \in s^{r'}$, $x \in s^r$, then $\{x',x\} \in Z$ holds iff $\{\pi_{j_1\cdots j_{r'}}(x'), \pi_{j_1\cdots j_r}(x)\} \in Z'$.

That is, Z' is determined by Z inside the classes $\overline{s}^r_{j_1\cdots j_r}$ and between two such classes one of whose index string extends the other, otherwise, for the "crossing" edges we impose no restriction.

Lemma 5. *There exists a $Z' \in \mathcal{Z}^n$ with* $\mathrm{FS}(Z') \subseteq \mathrm{FS}(X)$.

Proof. We have to show that there is some $Z' \in \mathcal{Z}^n$ in X. Our main concern is the disjointness proviso. Set $|s^r| = m_r - m_{r-1}$ (with $m_{-1} = 0$). Enumerate s^r as $s^r = \{a_i : m_{r-1} < i \leq m_r\}$. Let the formula $\varphi(x_1, \dots, x_{m_k})$ describe the graph $X|s$ on the first order theory of graphs, i.e., for $i < j$ it contains either the statement that $\{x_i, x_j\}$ is an edge, or the statement that it is not, according to if $\{a_i, a_j\} \in X$. We contract the string $x_{m_{r-1}+1} \cdots x_{m_r}$ of variables as $\overline{x}_r$, so that $\varphi(x_1, \dots, x_{m_k})$ becomes $\varphi(\overline{x}_0, \dots, \overline{x}_k)$. It suffices to show that for every natural number N

$$\exists \overline{x}_0 \exists^* \overline{x}_1 \cdots \exists^* \overline{x}_k \varphi(\overline{x}_0, \dots, \overline{x}_k)$$

is true in X where $\exists^*$ stands for "there exist N pairwise disjoint".

For $0 \leq i \leq k$ let $\varphi_i(\overline{x}_0, \dots, \overline{x}_i)$ denote the formula

$$\exists^* \overline{x}_{i+1} \cdots \exists^* \overline{x}_k \varphi(\overline{x}_0, \dots, \overline{x}_k).$$

We are going to prove by reverse induction that X satisfies $\varphi_i(\overline{a}_0, \dots, \overline{a}_i)$ for $0 \leq i \leq k$. This gives the required result for $i = 0$. For $i = k$ this is true, as it is just $\varphi(\overline{a}_0, \dots, \overline{a}_k)$. Assume that we have established $\varphi_{i+1}(\overline{a}_0, \dots, \overline{a}_{i+1})$ yet $\varphi_i(\overline{a}_0, \dots, \overline{a}_i)$ is false. That is, although we have $\varphi_{i+1}(\overline{a}_0, \dots, \overline{a}_{i+1})$ there are no N pairwise disjoint sets $\overline{b}$ with $\varphi_{i+1}(\overline{a}_0, \dots, \overline{a}_i, \overline{b})$. This implies that there is a finite set T such that if $\varphi_{i+1}(\overline{a}_0, \dots, \overline{a}_i, \overline{b})$ holds then $\overline{b}$ has a nonempty intersection with T, so, as $s^0 \cup \cdots \cup s^i \subseteq \delta_{i+1}$ there is such a T in δ_{i+1} but this contradicts that $s^{i+1} \cap \delta_{i+1} = \emptyset$. ∎

We are now going to construct a class $\mathcal{Z}$ of graphs. Set $\theta = \tau^\lambda$. Assume that the sets $\{W^\alpha_g : g : \alpha \to \theta, \alpha < \lambda\}$ are mutually disjoint, and we have the bijections $\pi^\alpha_g : [\delta_\alpha, \delta_{\alpha+1}) \to W^\alpha_g$. Every graph $Y \in \mathcal{Z}$ will have its vertex set $W = \bigcup\{W^\alpha_g : g : \alpha \to \theta, \alpha < \lambda\}$. We require that π^α_g be an isomorphism between $X|[\delta_\alpha, \delta_{\alpha+1})$ and $Y|W^\alpha_g$, and moreover, if $\alpha' < \alpha$, $g : \alpha \to \theta$, $g' = g|\alpha'$, then for $x' \in [\delta_{\alpha'}, \delta_{\alpha'+1})$, $x \in [\delta_\alpha, \delta_{\alpha+1})$, $\{x', x\} \in X$ holds iff $\{\pi^{\alpha'}_{g'}(x'), \pi^\alpha_g(x)\} \in Y$.

Lemma 6. *There is some $Y \in \mathcal{Z}$ with* $\mathrm{FS}(Y) \subseteq \mathrm{FS}(X)$.

Proof. By Lemma 5, for every finite $S \subseteq W$ there is a graph which is the restriction of some $Y \in \mathcal{Z}$ to S and is an induced subgraph of X. Now the Rado selection principle (i.e., compactness) gives the result. ∎

Lemma 7. *If $Y \in \mathcal{Z}$ then $P(Y,\tau,\kappa)$ holds.*

Proof. Assume that $Y \in \mathcal{Z}$. We define a partial coloring $F : Y \to \tau$ as follows. For every $\alpha < \lambda$, $g : \alpha \to \theta$, set $T_g = \bigcup\{W^{\beta}_{g|\beta} : \beta < \alpha\}$, and make sure that for every function $f : T_g \to \tau$ there is some extension $g' : \alpha+1 \to \theta$ of g, such that the following holds: if $x \in T_g$, $y \in W^{\alpha}_{g'}$, then, whenever $\{x,y\} \in Y$, then $F(\{x,y\}) = f(x)$. This can be done, as the number of these possible functions $f : T_g \to \tau$ is at most $\tau^{\lambda} = \theta$.

We claim that this (partial) coloring witnesses $P(Y,\tau,\kappa)$. Indeed, assume that $W = \bigcup\{W_i : i < \mu\}$ is a decomposition such that F does not assume some value $j(i) < \tau$ on $Y|W_i$. Choose a function $g : \lambda \to \theta$ with the property that if $\alpha' < \alpha < \lambda$, $x' \in W^{\alpha'}_{g|\alpha'}$, $x \in W^{\alpha}_{g|\alpha}$, then, if W_i, then $F(\{x',x\}) = j(i)$ holds. Such a function g exists, by our construction of F.

Now Y restricted to $\bigcup\{W^{\alpha}_{g|\alpha} : \alpha < \lambda\}$ is isomorphic to X, a μ^+-chromatic graph. All subgraphs of the form $Y|W^{\alpha}_{g|\alpha}$ have chromatic number at most μ. Y must have, therefore, an edge $\{x',x\}$ with $x' \in W^{\alpha'}_{g|\alpha'}$, $x \in W^{\alpha}_{g|\alpha}$, $\alpha' < \alpha < \lambda$, and that $x', x \in W_i$ for some $i < \mu$, but this is a contradiction, as $F(\{x',x\})$ gets the forbidden value, $j(i)$. ∎

To show the result for arbitrary κ we proceed by induction on κ. Again, let X be a graph with $\mathrm{Chr}(X) = \kappa$, and $\lambda = |X|$ be minimal. As before, consider a filtration into elementary submodels $V(X) = \bigcup\{V_\alpha : \alpha < \lambda\}$ of X. This splits the (edges of) X into two parts: if $X_\alpha = X \cap [V_{\alpha+1} - V_\alpha]^2$ for $\alpha < \lambda$, then let $X' = \bigcup\{X_\alpha : \alpha < \lambda\}$ and $X'' = X - X'$. If $\mathrm{Chr}(X'') = \kappa$, we proceed, as in the successor case. If $\mathrm{Chr}(X'') < \kappa$, then we clearly have $\mathrm{Chr}(X') = \kappa$. As we have $\kappa_\alpha = \mathrm{Chr}(X_\alpha) < \kappa$ for every $\alpha < \kappa$, this can only mean that $\sup\{\kappa_\alpha : \alpha < \lambda\} = \kappa$. Remove those V_α with $\kappa_\alpha \le \omega$ and V_1 in any case.

Then, for the remaining $\alpha < \lambda$ we can get a graph Z_α with $P(Z_\alpha, \tau, \kappa_\alpha)$ and $\mathrm{FS}(Z_\alpha) \subseteq \mathrm{FS}(X_\alpha)$. Again, Rado's selection principle gives that the vertex disjoint union of these graphs can be extended (via crossing edges) to a graph Y with $\mathrm{FS}(Y) \subseteq \mathrm{FS}(X)$ and clearly $P(Y,\tau,\kappa)$ holds.

So far we have proved that if $\mathrm{Chr}(X) = \kappa > \omega$ and τ is arbitrary then there is a graph Y with $P(Y,\tau,\kappa)$ and $\mathrm{FS}(Y) \subseteq \mathrm{FS}(X)$. To achieve equality in the latter set inequality we proceed as follows. Let $V' \subseteq V = V(X)$ be a countable elementary submodel, $V'' = V - V'$, $X'' = X|V''$. Notice that $\mathrm{Chr}(X'') = \kappa$. By the above, there is, on some set W disjoint from V, a graph Y with

$P(Y,\tau,\kappa)$ and $\mathrm{FS}(Y)\subseteq\mathrm{FS}(X'')$. Again, we can argue, using elementarity, and the Rado selection principle that there is a graph Z on $V\cup W$ that restrict to Y and X on W and V, respectively, with $\mathrm{FS}(Z)\subseteq\mathrm{FS}(X)$. As X itself is an induced subgraph of Z we must have equality here. ∎

In the last part of the paper we prove property P for some well-known graph constructions.

Theorem 8. *(a) If κ is an inaccessible cardinal or the successor of a regular cardinal then $P([\kappa]^2,\kappa,\kappa)$ holds.*
(b) If λ is a singular cardinal, then $P([\lambda^+]^2,\mu,\lambda^+)$ holds for every $\mu<\lambda$.

Proof. (a) The statement $P([\kappa^+]^2,\kappa^+,\kappa^+)$ for κ regular is a special case of Todorčević's result ([8]).

This easily implies the other claim; one can take $F:[\kappa]^2\to\kappa$ in such a way that if $\mu<\kappa$ is regular then the restriction of F to the pairs in $[\mu,\mu^+)$ witnesses $P([\mu^+]^2,\mu^+,\mu^+)$. Assume that $\kappa=\bigcup\{A_\varepsilon:\varepsilon<\tau\}$ is some partition of κ into $\tau<\kappa$ parts. By assumption, for every $\tau<\mu<\kappa$, μ regular, there is some $\varepsilon(\mu)<\tau$ that F assumes every ordinal $<\mu^+$ on $A_{\varepsilon(\mu)}\cap[\mu,\mu^+)$. As κ is regular, there is some $\varepsilon<\tau$ that $\{\mu<\kappa:\varepsilon(\mu)=\varepsilon\}$ is unbounded. But then, F assumes every ordinal below κ on A_ε.

(b) We can as well assume that $\mu>\omega$ is regular. Let

$$\{C_\alpha:\alpha<\lambda^+,\mathrm{cf}(\alpha)=\mu\}$$

be a club guessing sequence. Spelled out, this means that $C_\alpha\subseteq\alpha$ is a closed, unbounded subset of α of order type μ and if $E\subseteq\lambda^+$ is a closed, unbounded set then there is some α with $C_\alpha\subseteq E$. (The existence of this system is a result of Shelah, see [7], p. 132.) Assume that the increasing enumeration of C_α is $C_\alpha=\{c^\alpha_i:i<\mu\}$. Define $F:[\lambda^+]^2\to\mu$ in such a way that $F(x,\alpha)=i$ holds for $c^\alpha_i\le x<c^\alpha_{i+1}$. We show that F witnesses $P([\lambda^+]^2,\mu,\lambda^+)$. Assume it does not and that $\lambda^+=\bigcup\{A_\varepsilon:\varepsilon<\lambda\}$ is some decomposition with F omitting the color $i(\varepsilon)<\mu$ on A_ε. Define $T=\{\varepsilon<\lambda:\sup(A_\varepsilon)<\lambda^+\}$. Let $\xi<\lambda^+$ be large enough to be an upper bound for every $\sup(A_\varepsilon)$ ($\varepsilon\in T$). Let $E\subseteq\lambda^+$ be a closed, unbounded set with $\min(E)>\xi$ and with the property that if $[\gamma,\gamma')$ is a complementary interval of E then $[\gamma,\gamma')\cap A_\varepsilon\ne\emptyset$ holds for every $\varepsilon\notin T$. (The existence of E easily follows from closure arguments.) By assumption, there is some α with $C_\alpha\subseteq E$. We show that no A_ε can include α. Well, as $\xi<\min(E)\le\min(A_\varepsilon)$, surely α cannot be an element of an A_ε with $\varepsilon\in T$. Assume that $\alpha\in A_\varepsilon$ with some $\varepsilon\notin T$. Set $i=i(\varepsilon)$. As $\varepsilon\notin T$, there is some $c^\alpha_i\le x<c^\alpha_{i+1}$ with $x\in A_\varepsilon$. But then, by our construction, $F(x,\alpha)=i$, a contradiction. ∎

Theorem 9. *$P(\mathrm{Sh}_n(\exp_n(\kappa)),\kappa,\kappa^+)$ holds if $2\le n<\omega$ and κ is an infinite cardinal.*

Proof. By cardinal considerations it is possible to define a function

$$F:[\exp_n(\kappa)]^{n+1}\to\kappa$$

with the property that whenever $X\subseteq\exp_n(\kappa)$, $|X|\le\exp_{n-1}(\kappa)$, and $h:[X]^n\to\kappa$ then there exists a $y>\sup(X)$ such that

$$F(x_1,\dots,x_n,y)=h(x_1,\dots,x_n)\quad(x_1<\dots<x_n\in X).$$

We show that F witnesses $P(\mathrm{Sh}_n(\exp_n(\kappa)),\kappa,\kappa^+)$.

Assume indirectly that $g:[\exp_n(\kappa)]^n\to\kappa$, $t:\kappa\to\kappa$ are some functions with the property that if $x_0<\dots<x_n$ and

$$i=g(x_0,\dots,x_{n-1})=g(x_1,\dots,x_n)$$

then $F(x_0,\dots,x_n)\neq t(i)$.

Define the functions $g_n,\dots,g_1$ as follows. $g_n(x_1,\dots,x_n)=g(x_1,\dots,x_n)$. If g_{n-i+1} is already defined, set

$$g_{n-i}(x_{i+1},\dots,x_n)=\{g_{n-i+1}(x_i,x_{i+1},\dots,x_n):x_i<x_{i+1}\}\in P^i(\kappa),$$

where, as usual, $P(Y)$ is the power set of Y. Notice that the range of g_i has at most $\exp_{n-i}(\kappa)$ elements.

In what follows we are going to construct elements $x_i^{\overline{\alpha}}$ for some sequences $\overline{\alpha}=(\alpha_i,\dots,\alpha_n)$ of ordinals. If $\overline{\alpha}$ is longer, $\overline{\alpha}=(\alpha_j,\dots,\alpha_n)$ with $j<i$ our understanding is that $x_i^{\overline{\alpha}}=x_i^{\overline{\beta}}$ where $\overline{\beta}=(\alpha_i,\dots,\alpha_n)$.

Choose the elements $\{x_n^{\alpha_n}:\alpha<\exp_{n-1}(\kappa)\}$ in such a way that

$$\{g_1(x_n^{\alpha_n}):\alpha<\exp_{n-1}(\kappa)\}=\{g_1(y):\alpha<\exp_n(\kappa)\}.$$

If $\overline{\alpha}=(\alpha_{i+1},\dots,\alpha_n)$ and the string $\overline{x}=(x_{i+1}^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}})$ are determined, select the elements $\{x_i^{\alpha_i,\overline{\alpha}}:\alpha_i<\exp_{i-1}(\kappa)\}$ in such a way that

$$\{g_{n-i+1}(x_i^{\alpha_i,\overline{\alpha}},\overline{x}):\alpha_i<\exp_{i-1}(\kappa)\}=\{g_{n-i+1}(x,\overline{x}):x<x_{i+1}^{\overline{\alpha}}\}.$$

Put

$$X=\{x_i^{\overline{\alpha}}:1\le i\le n,\alpha_i<\exp_{i-1}(\kappa)\}$$

a set of cardinal $\exp_{n-1}(\kappa)$. Define h on X with

$$h(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}})=t(g(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}})).$$

By our assumption on F there is an element $y > \sup(X)$ such that for every string $\overline{\alpha}$ we have $F(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}},y)=h(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}})$.

To conclude, we argue as follows.

There is some $x_n^{\alpha_n}$ that $g_1(x_n^{\alpha_n})=g_1(y)$.

There is some $x_{n-1}^{\alpha_{n-1},\alpha_n}$ that $g_1(x_{n-1}^{\alpha_{n-1},\alpha_n},x_n^{\alpha_n})=g_1(x_n^{\alpha_n},y)$.

Continuing, we eventually get the string $(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}})$ with

$$g(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}}) = g(x_2^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}},y).$$

And then

$$F(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}},y) = h(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}}) = t(g(x_1^{\overline{\alpha}},\dots,x_n^{\overline{\alpha}}))$$

a contradiction. ∎

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